Algebra 3/4 Learning Progression

Summary of Year

Building on their work with linear, quadratic, and exponential functions, students extend their repertoire of functions to include polynomial, rational, and radical functions. Students work closely with the expressions that define the functions, and continue to expand and hone their abilities to model situations and to solve equations, including solving quadratic equations over the set of complex numbers and solving exponential equations using the properties of logarithms.

The Mathematical Practice Standards apply throughout each course and, together with the content standards, prescribe that students experience mathematics as a coherent, useful, and logical subject that makes use of their ability to make sense of problem situations.

Recommended Fluencies for Algebra II

- Divide polynomials with remainder by inspection in simple cases.
- See structure in expressions and use this structure to rewrite expressions (e.g., factoring, grouping).
- Translate between recursive definitions and closed forms for problems involving sequences and series.

Unit 1A/B: Polynomial, Rational, and Radical Relationships
Unit 2: Trigonometric Functions
Unit 3A: Exponential Functions
Unit 3B: Logarithmic Functions
Unit 4: Inferences and Conclusions from Data

WSLS (CCSS) Major Emphasis Clusters
The Real Number System • Extend the properties of exponents to rational exponents Seeing Structure in Expressions • Interpret the structure of expressions • Write expressions in equivalent forms to solve problems Arithmetic with Polynomials and Rational Expressions • Understand the relationship between zeros and factors of polynomials Reasoning with Equations and Inequalities • Understand solving equations as a process of reasoning and explain the reasoning • Represent and solve equations and inequalities graphically Interpreting Functions • Interpret functions that arise in applications in terms of the context Building Functions • Build a function that models a relationship between two quantities Making Inferences and Justifying Conclusions • Make inferences and justify conclusions from sample surveys, experiments and observational studies

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Rationale for Unit Sequence in Algebra 3-4

Unit 1A: In this unit students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations. The role of factoring, as both an aid to the algebra and to the graphing of polynomials, is explored. Students continue to build upon the reasoning process of solving equations as they solve polynomial, rational, and radical equations, as well as linear and non-linear systems of equations. Students use appropriate tools to analyze the key features of a graph or table of a polynomial function and relate those features back to the two quantities in the problem that the function is modeling.

Unit 1B: Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers. Students identify zeros of polynomials and make connections between zeros of polynomials and solutions of polynomial equations. The unit culminates with the fundamental theorem of algebra as the ultimate result in factoring.

Unit 2: Building on their previous work with functions, and on their work with trigonometric ratios and circles in Geometry, students extend trigonometric functions to all (or most) real numbers. To reinforce their understanding of these functions, students begin building fluency with the values of sine, cosine, and tangent at $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, etc. Students make sense of periodic phenomena as they model with trigonometric functions. They identify the periodicity, midline, and amplitude from graphs of data and use them to construct sinusoidal functions that model situations from both the biological and physical sciences. They extend the concept of polynomial identities to trigonometric identities and prove simple trigonometric identities such as the Pythagorean identity; these identities are then used to solve problems.

(Instructional Note: Unit 2 should be finished by the end of semester. Since these standards are additional clusters, less than 5% of your instructional time should be spent on these standards. Feel free to cut if necessary. Unit 3 should begin at the start of 2nd semester).

Unit 3A: In this unit, students synthesize and generalize what they have learned about a variety of function families. They extend the domain of exponential functions to the entire real line. They explore (with appropriate tools) the effects of transformations on graphs of diverse functions, including functions arising in an application. They notice, by looking for general methods in repeated calculations, that the transformations on a graph always have the same effect regardless of the type of the underlying function. These observations lead to students to conjecture and construct general principles about how the graph of a function changes after applying a function transformation to that function.

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Students identify appropriate types of functions to model a situation, they adjust parameters to improve the model, and they compare models by analyzing appropriateness of fit and making judgments about the domain over which a model is a good fit. The description of modeling as, "*the process of choosing and using mathematics and statistics to analyze empirical situations, to understand them better, and to make decisions,*" is at the heart of this unit. In particular, through repeated opportunities in working through the modeling cycle (see page 61 of the CCLS), students acquire the insight that the same mathematical or statistical structure can sometimes model seemingly different situations.

Unit 3B: In this unit, students synthesize and generalize what they have learned about a variety of function families. They extend their work with exponential functions to include solving exponential equations with logarithms. They explore (with appropriate tools) the effects of transformations on graphs of diverse functions, including functions arising in an application. They notice that the transformations on a graph of a logarithmic function relate to the logarithmic properties. By looking for general methods in repeated calculations, the transformations on a graph always have the same effect regardless of the type of the underlying function. These observations lead to students to conjecture and construct general principles about how the graph of a function changes after applying a function transformation to that function.

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Unit 4: In this unit, students see how the visual displays and summary statistics they learned in earlier grades relate to different types of data and to probability distributions. They identify different ways of collecting data, including sample surveys, experiments, and simulations, and the role that randomness and careful design play in the conclusions that can be drawn. Students create theoretical and experimental probability models following the modeling cycle (see page 61 of CCLS). They compute and interpret probabilities from those models for compound events, attending to mutually exclusive events, independent events, and conditional probability.

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Alignment Chart

Unit and Approximate Number of Instructional Days	Washington State Learning Standards Addressed in Algebra 3/4 Units
Unit 1A: **Polynomial, Rational, and Radical Relationships** (23 days) SMP 1,2,3,4,5,6,7,8	*Summarize, represent, and interpret data on two categorical and quantitative variables* S.ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related. a. Fit a function to the data; use functions fitted to data to solve problems in the context of the data. *Use given functions or choose a function suggested by the context. Emphasize linear and exponential models.* b. Informally assess the fit of a function by plotting and analyzing residuals. c. Fit a linear function for a scatter plot that suggests a linear association. *Analyze functions using different representations* F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ a. Graph linear and quadratic functions and show intercepts, maxima, and minima. b. ~~Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions~~. c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior. d. (+) Graph rational functions, identifying zeros ~~and asymptotes~~ when suitable factorizations are available, ~~and showing end behavior~~. e. ~~Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude~~.

	Reason quantitatively and use units to solve problems N.Q.2 Define appropriate quantities for the purpose of descriptive modeling. *Perform arithmetic operations with complex numbers* N.CN.1 Know there is a complex number i such that $i^2 = –1$, and every complex number has the form $a + bi$ with a and b real. N.CN.2 Use the relation $i^2 = –1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers. *Use complex numbers in polynomial identities and equations* N.CN.7 Solve quadratic equations with real coefficients that have complex solutions. *Understand solving equations as a process of reasoning and explain the reasoning* A.REI.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. *Solve equations and inequalities in one variable* A.REI.4 Solve quadratic equations in one variable. a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form. b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the

	initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b. *Additional clusters:* *Solve systems of equations* A.REI.6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$.*
Unit 1B: Polynomial, Rational, and Radical Relationships (25 days) SMP 1,2,3,4,5,6,7,8	*Interpret the structure of expressions* A.SSE.2 Use the structure of an expression to identify ways to rewrite it. *For example, see $x^4 - y^4$ as $(x^2)^2 - (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 - y^2)(x^2 + y^2)$.* *Understand the relationship between zeros and factors of polynomials* A.APR.2 Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$. A.APR.3 Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

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	Use polynomial identities to solve problems A.APR.4 Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.* *Rewrite rational expressions* A.APR.6 Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system. *Understand solving equations as a process of reasoning and explain the reasoning* A.REI.1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. A.REI.2 Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
Unit 2: **Trigonometric Functions** (17 days) SMP 1,2,3,4,5,6,7,8 (Instructional Note: Unit 2 should be finished by the end of semester. Since these standards are additional clusters, less than	*Extend the domain of trigonometric functions using the unit circle* F.TF.1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle. F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

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5% of your instructional time should be spent on these standards. Feel free to cut if necessary. Unit 3 should begin at the start of 2nd semester).	*Model periodic phenomena with trigonometric functions* F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★ *Prove and apply trigonometric identities* F.TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$,or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.
Unit 3A: **Exponential Functions** (25 days) SMP 1,2,3,4,5,6,7,8	*Extend the properties of exponents to rational exponents* N.RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.* N.RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents. N.RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational. *Reason quantitatively and use units to solve problems* N.Q.2 Define appropriate quantities for the purpose of descriptive modeling.

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	Write expressions in equivalent forms to solve problems A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. ★ c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.* A.SSE.4 Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. *For example, calculate mortgage payments.*★ *Create equations that describe numbers or relationships* A.CED.1 Create equations and inequalities in one variable and use them to solve problems. *Include ~~equations arising from linear and quadratic functions, and simple rational and~~ exponential functions.* *Represent and solve equations and inequalities graphically* A.REI.11 Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, ~~and logarithmic functions~~.★

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	Understand the concept of a function and use function notation F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *For example, the Fibonacci sequence is defined recursively by $f(0) = f(1) = 1$, $f(n+1) = f(n) + f(n-1)$ for $n \geq 1$.* *Interpret functions that arise in applications in terms of the context* F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★ F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★ *Analyze functions using different representations* F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ e. Graph exponential ~~and logarithmic~~ functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

	F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions. *For example, identify percent rate of change in functions such as* $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, *and classify them as representing exponential growth or decay.* F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* *Build a function that models a relationship between two quantities* F.BF.1 Write a function that describes a relationship between two quantities.★ a. Determine an explicit expression, a recursive process, or steps for calculation from a context. b. Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* F.BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.★ *Build new functions from existing functions* F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

	F.BF.4 Find inverse functions. a. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. *For example, f(x) =2 x^3 or f(x) = (x+1)/(x–1) for x ≠1.* b. (+) Verify by composition that one function is the inverse of another. c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse. d. (+) Produce an invertible function from a non-invertible function by restricting the domain. *Construct and compare linear and exponential models and solve problems* F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table). F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where *a*, *c*, and *d* are numbers and the base *b* is 2, 10, or *e*; evaluate the logarithm using technology. *Interpret expressions for functions in terms of the situation they model* F.LE.5 Interpret the parameters in ~~a linear or~~ (*an*) exponential function in terms of a context.
Unit 3B: **Logarithmic Functions** (20 days) SMP 1,2,3,4,5,6,7,8	*Reason quantitatively and use units to solve problems* N.Q.2 Define appropriate quantities for the purpose of descriptive modeling.

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	Represent and solve equations and inequalities graphically A.REI.11 Explain why the *x*-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, ~~exponential~~, and logarithmic functions.★ *Interpret functions that arise in applications in terms of the context* F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*★ Including logarithmic functions. F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ *Analyze functions using different representations* F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ e. Graph ~~exponential and~~ logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude. F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a*

	graph of one quadratic function and an algebraic expression for another, say which has the larger maximum. *Build new functions from existing functions* F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.* F.BF.4 Find inverse functions. a. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse. *For example,* $f(x) = 2\,x^3$ *or* $f(x) = (x+1)/(x–1)$ *for* $x \neq 1$. b. (+) Verify by composition that one function is the inverse of another. c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse. d. (+) Produce an invertible function from a non-invertible function by restricting the domain. *Construct and compare linear and exponential models and solve problems* F.LE.4 For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.
Unit 4: **Inferences and Conclusions from Data** (40 days)	*Summarize, represent, and interpret data on a single count or measurement variable* S.ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which

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SMP 1,2,3,4,5,6,7,8	such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve. *Understand and evaluate random processes underlying statistical experiments* S.IC.1 Understand statistics as a process for making inferences about population parameters based on a random sample from that population. S.IC.2 Decide if a specified model is consistent with results from a given data-generating process, e.g., using simulation. *For example, a model says a spinning coin falls heads up with probability 0.5. Would a result of 5 tails in a row cause you to question the model?* *Make inferences and justify conclusions from sample surveys, experiments, and observational studies* S.IC.3 Recognize the purposes of and differences among sample surveys, experiments, and observational studies; explain how randomization relates to each. S.IC.4 Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling. S.IC.5 Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant. S.IC.6 Evaluate reports based on data. *Understand independence and conditional probability and use them to interpret data* S.CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events (“or,” “and,” “not”).

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	S.CP.2 Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent. S.CP.3 Understand the conditional probability of *A* given *B* as *P*(*A* and *B*)/*P*(*B*), and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*. S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results.* S.CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer.* *Use the rules of probability to compute probabilities of compound events in a uniform probability model* S.CP.6 Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model. S.CP.7 Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model.